Thoughts on Matrices for A30 & B30

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Systems of Equations -- Math A30

When matrices are introduced in high school we must avoid the tendency to head straight towards finding inverses and using some formula to solve systems of linear equations. As usual, teaching by formulas is very often unsuccessful. To simply give students formulas to calculate the inverse of a 2 by 2 matrix etc. is not teaching them mathematics, quite the contrary. Matrices should be introduced mainly as a notational convenience. Those students who master the relevant techniques for solving systems of equations with matrices and who subsequently take Math B30 will be ready to solve systems of equations using inverses.

Why matrices? Take a simple problem with two equations and two unknowns and solve in the usual manner.

3x - 6y = 9

2x + 5y = 4

multiply eqn 1 by 1/3:

x - 2y = 3

2x + 5y = 4

take two of new eqn 1 from eqn 2:

x - 2y = 3

0x + 9y = -2

multiply new eqn 2 by 1/9:

x - 2y = 3

0x + 1y = -2/9

add two new eqn 2's to new eqn 1:

x - 0y = 23/9

0x + 1y = -2/9

i.e. x = 23/9 and y = -2/9.

Now introduce a rectangular array, do the work in a parallel fashion to motivate the defining of the matrix. (The present medium does not allow us to conveniently use matrix notation without introducing graphics into the document so for the present note we will use

$$\left[\begin{array}{cc|c} 3 & -6 & 9 \\ 2 & 5 & 4 \end{array}\right]$$

for the matrix $\left[\begin{array}{cc|c} 3 & -6 & 9 \\ 2 & 5 & 4 \end{array}\right]$ etc.) For our problem, we then have

$$\left[\begin{array}{cc|c} 3 & -6 & 9 \\ 2 & 5 & 4 \end{array}\right]$$

where we have x, y & constants columns respectively;

multiply row 1 by 1/3:

$$\left[\begin{array}{cc|c} 1 & -2 & 3 \\ 2 & 5 & 4 \end{array}\right]$$

take two of new row 1 from row 2:

$$\left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 9 & -2 \end{array}\right]$$

multiply new row 2 by 1/9:

$$\left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 1 & -2/9 \end{array}\right]$$

add two new row 2's to new row 1:

$$\left[\begin{array}{cc|c} 1 & 0 & 23/9 \\ 0 & 1 & -2/9 \end{array}\right]$$

and finally, we read this as x = 23/9 and y = -2/9.

We have solved the system twice in exactly the same way but in the second solution we've avoided about 10 equal signs and about the same number of + or - signs. This is simply a good notation that speeds up the work. We may now define such rectangular arrays as matrices.

Solve a number of systems this way so that students get used to the operations on a matrix - these are called row operations and at this level that's all that we should be concentrating on. Using a matrix in this manner is certainly not confusing for students as it is related to something they know already -- the solving of systems of equations.

We can now introduce the problem of simultaneously solving families of systems of equations such as solving

3x - 6y = 9 and 2x + 5y = 4

together with solving

3x - 6y = 6 and 2x + 5y = 3,

where only the right hand sides of these systems differ. Look at a matrix solution to both

$$\left[\begin{array}{cc|c} 3 & -6 & 9 \\ 2 & 5 & 4 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -2 & 3 \\ 2 & 5 & 4 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 9 & -2 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 1 & -2/9 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 23/9 \\ 0 & 1 & -2/9 \end{array}\right]$$

i.e. x = 23/9 & y = -2/9

and

$$\left[\begin{array}{cc|c} 3 & -6 & 6 \\ 2 & 5 & 3 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -2 & 2 \\ 2 & 5 & 3 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 9 & -1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 1 & -1/9 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 16/9 \\ 0 & 1 & -1/9 \end{array}\right]$$

i.e. x = 16/9 & y = -1/9

One should point out the obvious duplication of effort and ask the students to consider a family of half a dozen such systems (rather than two) where the only difference is the constant column on the right hand side. Obviously there's a huge duplication of effort involved in solving them all. (This is an important observation and a good motivation to introduce the concept of a matrix inverse later in Math B30.)

$$\left[\begin{array}{cc|cc} 3 & -6 & 9 & 6 \\ 2 & 5 & 4 & 3 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & -2 & 3 & 2 \\ 2 & 5 & 4 & 3 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & -2 & 3 & 2 \\ 0 & 9 & -2 & -1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & -2 & 3 & 2 \\ 0 & 1 & -2/9 & -1/9 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 23/9 & 16/9 \\ 0 & 1 & -2/9 & -1/9 \end{array}\right]$$

simultaneously yielding:

x = 23/9 & y = -2/9 and x = 16/9 & y = -1/9

Clearly the above can be extended to more than two simultaneous systems.

Matrices in Math B30 - Inverses

Once you have introduced matrix multiplication in your class you can easily extend the above idea to finding an matrix inverse. We will denote the inverse of a matrix A by A^-1 so that if I represents the identity marix, A(A^-1) = (A^-1)A = I. The important idea is that if we have a system of equations, say

ax + by = e

cx + dy = f

we let the matrices A, X and B represent respectively

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } \begin{bmatrix} e \\ f \end{bmatrix}$$

so that the system of equations in matrix form is AX = B and by premultiplying by the inverse of A we have (A^-1)AX = IX = X = (A^-1)B i.e. to find the unknown matrix X we need only premultiply B by A inverse.

Thus we would like to find a way to obtain the inverse of a square matrix A. Let's look at the 2 x 2 case -- this is identical to the n x n case.

To this end we will use as a matrix A that which arose from the systems we looked at earlier. Ask the students to solve

3x - 6y = 1 and 2x + 5y = 0

simultaneously with solving

3x - 6y = 0 and 2x + 5y = 1,

i.e.

$$\left[\begin{array}{cc|cc} 3 & -6 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & -2 & 1/3 & 0 \\ 2 & 5 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & -2 & 1/3 & 0 \\ 0 & 9 & -2/3 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & -2 & 1/3 & 0 \\ 0 & 1 & -2/27 & 1/9 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 5/27 & 2/9 \\ 0 & 1 & -2/27 & 1/9 \end{array}\right]$$

Now we get to the interesting point. Have the students take the right hand part of this matrix and multiply by the constant columns of the systems we looked at earlier (above).

$$\begin{bmatrix} 5/27 & 2/9 \\ -2/27 & 1/9 \end{bmatrix} \begin{bmatrix} 9 \\ 4 \end{bmatrix} = \begin{bmatrix} 23/9 \\ -2/9 \end{bmatrix}$$

and

$$\begin{bmatrix} 5/27 & 2/9 \\ -2/27 & 1/9 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 16/9 \\ -1/9 \end{bmatrix}$$

Students should recognize their earlier solutions as the result. Now have them do the multiplications

$$\begin{bmatrix} 5/27 & 2/9 \\ -2/27 & 1/9 \end{bmatrix} \begin{bmatrix} 3 & -6 \\ 2 & 5 \end{bmatrix}$$

and

$$\begin{bmatrix} 3 & -6 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 5/27 & 2/9 \\ -2/27 & 1/9 \end{bmatrix}$$

In both cases we obtain the 2 x 2 identity matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so that indeed we have produced the inverse of our matrix A as the right hand part of the matrix we produced to solve the two systems in this section. Now is a good time to talk about properties of the inverse of a square matrix. Show them how once we know the inverse of A we can quickly solve AX = B for any number of B's we choose.
