Further Properties Assigned to arcsine and arccosine

Gregory V. Akulov, teacher, Luther College High School, Regina, Saskatchewan
Oleksandr G. Akulov, MM in OR, BMath, Vancouver, British Columbia

Comprehensive Handbook of Mathematics[1] lists special class of piecewise identities that includes the following two:

$$\sin^{-1}x+\sin^{-1}y=\begin{cases}\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right), & \text{if } xy\le 0, \textbf{ or } x^2+y^2\le 1\\ \pi-\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right), & \text{if } x>0, y>0, \textbf{ and } x^2+y^2>1\\ -\pi-\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right), & \text{if } x<0, y<0, \textbf{ and } x^2+y^2>1\end{cases},$$

$$\cos^{-1}x+\cos^{-1}y=\begin{cases}\cos^{-1}\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right), & \text{if } x+y\ge 0\\ 2\pi-\cos^{-1}\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right), & \text{if } x+y<0\end{cases}.$$

The alternative, non-piecewise identities advancing the list

$$\sin^{-1}x+\sin^{-1}y=2\sin^{-1}v\,, \tag{1}$$

$$\cos^{-1}x+\cos^{-1}y=2\cos^{-1}v\,, \tag{2}$$

$$\text{where } 2v=\sqrt{(1+x)(1+y)}-\sqrt{(1-x)(1-y)}\,, \tag{a}$$

were discovered in Canada in 2010. The same year their algebraic component (a) prompted conjugate invention of arc midpoint computation[2]. Note, that (1), (2), (a) is a complete version of (1), (2) with

$$v=\frac{x+y}{\sqrt{2\left(1+xy+\sqrt{(1-x^2)(1-y^2)}\right)}}\,, \tag{b}$$

that was revealed in Ukraine in 1998. Beside additional simplicity, version (a) works at any point of the 2x2 square $x, y\in[-1,1]$, while version (b) fails at some of its points. As an exercise one…

I. Determine the points in $x, y\in[-1,1]$ at which* (1), (2), (b) *does not work.

It is easy to derive (2) from (1)

$$\cos^{-1}x+\cos^{-1}y=\frac{\pi}{2}-\sin^{-1}x+\frac{\pi}{2}-\sin^{-1}y=\pi-(\sin^{-1}x+\sin^{-1}y)$$

$$=\pi-2\sin^{-1}v=2\left(\frac{\pi}{2}-\sin^{-1}v\right)=2\cos^{-1}v\,.$$

The formal logic, however, requires the direct proof of (1). As an exercise two…

***II. Prove* (1), (a) *directly, without using* (2).**

IB students may find this topic informative for Mathematics explorations and essays in Theory of Knowledge.

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[1] Bronshtein, I.N., K.A. Semendyaev, G. Musiol, and H. Muehlig. *Handbook of Mathematics*. Fifth Edition ed. New York: Springer, 2007. p. 86

[2] http://mathcentral.uregina.ca/RR/database/RR.09.10/akulov2.html